Balanced Quantum-Like Model for Decision Making

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Abstract. Clues from psychology indicate that human cognition is not only based on classical probability theory as explained by Kolmogorov's axioms but additionally on quantum probability. We explore the relation between the law of total probability and its violation resulting in the law of total quantum probability. The violation results from an additional interference that influences the classical probabilities. Outgoing from this exploration we introduce a balanced Bayesian quantum-like model that is based on probability waves. The law of maximum uncertainty indicates how to choose a possible phase value of the wave resulting in a meaningful probability value.

Keywords: Quantum Cognition; Law of Total Probability; Probability Waves; Decision Making.

1 Introduction

Clues from psychology indicate that human cognition is not only based on traditional probability theory as explained by Kolmogorov's axioms but additionally on quantum probability [8, 6, 7, 5, 4, 14]. For example, humans when making decisions violate the law of total probability. The emerging field that studies the corresponding models is called quantum cognition. In this work, we introduce a balanced Bayesian quantum-like model that is based on probability waves. The law of maximum uncertainty indicates how to choose a possible phase value of the wave resulting in a meaningful probability value. We demonstrate the model and the law on several experiments of the literature concerned the prisoner's dilemma game and the two stage gambling game. We compare the results with previous works that deal with predictive quantum-like models for decision making. The results obtained show that the model can make predictions regarding human decision-making with a meaningful interpretation.

1.1 Prisoner's Dilemma Game and Probability waves

In the prisoner's dilemma game, there are two prisoners, prisoner x and prisoner y. They have no means of communicating with each other. Each prisoner is offered by the prosecutors a bargain: by testifying against the other one she can betray the other one (Defect). On the other hand, the prisoner can refuse the deal and cooperate with the other one by remaining silent [18].

Several psychological experiments were made assuming that the probability of prisoner x cooperating is $p(x) = 0.5$ and the probability of defecting is $p(\neg x) = 0.5$. The participants of the experiment were asked three different questions.

- What is the probability that the prisoner y defects given x defects, $p(\neg y|\neg x)$.
- What is the probability that the prisoner y defects given x cooperates, $p(\neg y|x)$.
- What is the probability that the prisoner y defects given there is no information present about knowing if prisoner x cooperates or defects. This can be expressed by

$$p(\neg y) = p(\neg y, x) + p(\neg y, \neg x) = p(\neg y|x) \cdot p(x) + p(\neg y|\neg x) \cdot p(\neg x). \qquad (1)$$

  This relationship can be represented by a graph (see Figure 1) that indicates the influence between events x and y.

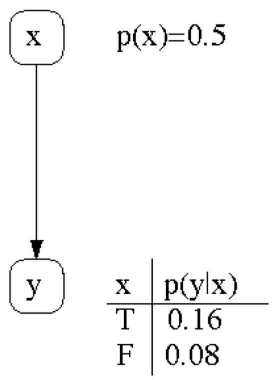

Fig. 1. The causal relation between events x and y represented by a direct graph of two nodes. Note that each node is followed by a conditional probability table that specifies the probability distribution of that node according to its parent node. This direct graph of two nodes representation corresponds to a simple Bayesian network.

In Table 1, we summarise the results of several experiments of the literature concerned with the prisoner's dilemma experiment.

Table 1. Experimental results obtained in four different works of the literature for the prisoner's dilemma game. The column $p(\neg y|\neg x)$ corresponds to the probability of $defecting$ given that it is known that the other participant chose to $defect$. The column $p(\neg y|x)$ corresponds to the probability of $defecting$ given that it is known that the other participant chose to *cooperate*. Finally, the column $p_{sub}(\neg y)$ corresponds to the subjective probability of the second participant choosing the $defect$ action given there is no information present about knowing if prisoner x cooperates or defects. The column $p(\neg y)$ corresponds to the classical probability.

Experiment	$p(\neg y\|\neg x)$	$p(\neg y\|x)$	$p_{sub}(\neg y)$	$p(\neg y)$
(a) [19]	0.97	0.84	0.63	0.9050
(b) [17]	0.82	0.77	0.72	0.7950
(c) [3]	0.91	0.84	0.66	0.8750
(d) [10]	0.97	0.93	0.88	0.9500
(e) Average	0.92	0.85	0.72	0.8813

1.2 Two Stage Gambling Game

In the two stage gambling game the participants were asked whether they want to play a gamble that has an equal chance of winning $p(x) = 0.5$ or losing $p(\neg x) = 0.5$ [18]. The participants of the experiment were asked three different questions.

- What is the probability that they play the gamble y if they had lost the first gamble x, $p(y|\neg x)$
- What is the probability that they play the gamble y if they had won the first gamble x, $p(y|x)$
- What is the probability that they play the gamble y given there is no information present knowing if they had won the first gamble x. This would by by the law of total probability

$$p(y) = p(y|x) \cdot p(x) + p(y|\neg x) \cdot p(\neg x) \quad (2)$$

In Table 2, we summarise the results of several experiments of the literature concerned with the two stage gambling game.

2 Quantum Probabilities and Waves

Beside quantum cognition quantum physics was the only branch in science that evaluated a probability $p(x)$ an state x as the mode-square of a probability amplitude $A(x)$ represented by a complex number

$$p(x) = |A(x)|^2 = \|A(x)\|^2 = A(x)^* \cdot A(x). \quad (3)$$

This is because the product of complex number with its conjugate is always a real number. With

$$A(x) = \alpha + \beta \cdot i \quad (4)$$

Table 2. Experimental results obtained in three different works of the literature indicating the probability of a player choosing to make a second gamble for the two stage gambling game. The column $p(y|\neg x)$ corresponds to the probability when the outcome of the first gamble is known to be lose. The column $p(y|x)$ corresponds to the probability when the outcome of the first gamble is known to be win. Finally, the column $p_{sub}(y)$ corresponds to the subjective probability when the outcome of the first gamble is not known. The column $p(y)$ corresponds to the classical probability.

Experiment	$p(y\|\neg x)$	$p(y\|x)$	$p_{sub}(y)$	$p(y)$
(i) [20]	0.58	0.69	0.37	0.6350
(ii) [15]	0.47	0.72	0.48	0.5950
(iii) [16]	0.45	0.63	0.41	0.5400
(iv) Average	0.50	0.68	0.42	0.5900

$$A(x)^* \cdot A(x) = (\alpha - \beta \cdot i) \cdot (\alpha + \beta \cdot i) = \alpha^2 + \beta^2 = |A(x)|^2. \tag{5}$$

Quantum physics by itself does not offer any justification or explanation beside the statement that it just works fine [2]. We can map the classical probabilities into amplitudes using the polar coordinate representation

$$a(x, \theta_1) = \sqrt{p(x)} \cdot e^{i \cdot \theta_1} = A(x), \quad a(y, \theta_2) = \sqrt{p(y)} \cdot e^{i \cdot \theta_2} = A(y). \tag{6}$$

The amplitudes represented in polar coordinate form contains a new free parameter θ, which corresponds to the phase of the wave.

2.1 Intensity Waves

The intensity wave is defined as

$$I(y, \theta_1, \theta_2) = |a(y, x, \theta_1) + a(y, \neg x, \theta_2)|^2 \tag{7}$$

$$I(y, \theta_1, \theta_2) = p(y, x) + p(y, \neg x) + 2 \cdot \sqrt{p(y, x) \cdot p(y, \neg x)} \cdot \cos(\theta_1 - \theta_2) \tag{8}$$

Note that for simplification we can replace $\theta_1 - \theta_2$ with θ,

$$\theta = \theta_1 - \theta_2$$

$$I(y, \theta) = p(y) + 2 \cdot \sqrt{p(y, x) \cdot p(y, \neg x)} \cdot \cos(\theta) \tag{9}$$

and

$$I(\neg y, \theta_{\neg 1}, \theta_{\neg 2}) = |a(\neg y, x, \theta_{\neg 1}) + a(\neg y, \neg x, \theta_{\neg 2})|^2 \tag{10}$$

with

$$\theta_{\neg} = \theta_{\neg 1} - \theta_{\neg 2}$$

$$I(\neg y, \theta_{\neg}) = p(\neg y) + 2 \cdot \sqrt{p(\neg y, x) \cdot p(\neg y, \neg x)} \cdot \cos(\theta_{\neg}) \tag{11}$$

and for certain phase values

$$I(y, \theta) + I(\neg y, \theta_{\neg}) \neq 1. \tag{12}$$

In Figure 2 a) we see two intensity waves in relation to the phase with the parametrisation as indicated in corresponding to the values of Figure 1 corresponding to the Table 1 values in (e).

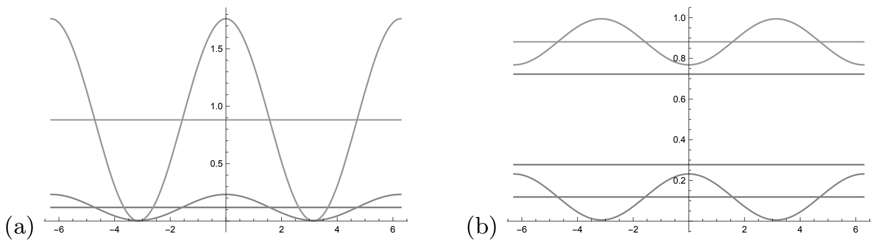

Fig. 2. (a) Two intensity waves $I(y, \theta)$, $I(\neg y, \theta_\neg)$ in relation to the phase $(-2 \cdot \pi, 2 \cdot \pi)$ with the parametrisation as indicated in corresponding to the values of Figure 1. Note that the two waves oscillate around $p(y) = 0.1950$ and $p(\neg y) = 0.8050$ (the two lines). (b) The resulting probability waves as determined by the law of balance, the bigger wave is replaced by the negative smaller one.

2.2 The Law of Balance and Probability Waves

Intensity waves $I(y, \theta)$ and $I(\neg y, \theta_\neg)$ are probability waves $p(y, \theta)$ and $p(\neg y, \theta_\neg)$ if:

1. they are positive
$$0 \leq p(y, \theta), \quad 0 \leq p(\neg y, \theta_\neg); \tag{13}$$
2. they sum to one
$$p(y, \theta) + p(\neg y, \theta_\neg) = p(y) + p(\neg y) = 1; \tag{14}$$
3. they are smaller or equal to one
$$p(y, \theta) \leq 1, \quad p(\neg y, \theta_\neg) \leq 1. \tag{15}$$

Simply speaking, the law states that the bigger wave is replaced by a smaller negative one.

Probability Waves are Positive Since the norm is being positive or more precisely non-negative, we can represent a quadratic form by l_2 norm

$$(a(x, \theta_1) + a(y, \theta_2) \cdot a(x, \theta_1) + a(y, \theta_2)) = \|a(x, \theta_1) + a(y, \theta_2)\|^2$$

and it follows

$$0 \leq \|a(x, \theta_1) + a(y, \theta_2)\|^2.$$

Probability Waves Sum to one according by the Law of Balance Instead of simple normalisation of the intensity we propose the law of balance. The interference is balanced, which means that the interference of $p(y, \theta)$ and $p(\neg y, \theta_\neg)$ cancel each out.

$$\sqrt{p(y, x) \cdot p(y, \neg x)} \cdot \cos(\theta) = -\sqrt{p(\neg y, x) \cdot p(\neg y, \neg x)} \cdot \cos(\theta_\neg). \tag{16}$$

We can solve the Equation for the phase $\theta_\neg$ or θ resulting in two possible cases. For $\theta_\neg$ we get

$$\theta_\neg = \cos^{-1}\left(-\sqrt{\frac{p(y|x)\cdot p(y|\neg x)}{p(\neg y|x)\cdot p(\neg y|\neg x)}}\cdot\cos(\theta)\right) \tag{17}$$

Since $\cos^{-1}(x)$ is only defined for $x \in [-1,1]$ the Equation 21 is valid for the constraint

$$\frac{p(y|x)\cdot p(y|\neg x)}{p(\neg y|x)\cdot p(\neg y|\neg x)} \leq 1. \tag{18}$$

corresponding to

$$p(y) \leq p(\neg y). \tag{19}$$

Since

$$p(y|x)\cdot p(y|\neg x) \leq p(\neg y|x)\cdot p(\neg y|\neg x)$$

$$p(y|x)\cdot p(y|\neg x) \leq (1-p(y|x))\cdot(1-p(y|\neg x))$$

$$0 \leq 1-p(y|x)-p(y|\neg x).$$

$p(y) \leq p(\neg y)$ is equivalent for $p(x) = p(\neg x)$ to

$$p(y|x)+p(y|\neg x) \leq (1-p(y|x))+(1-p(y|\neg x))$$

$$0 \leq 1-p(y|x)-p(y|\neg x).$$

The smaller probability wave determines the other probability wave as

$$p(\neg y,\theta_\neg) = 1-p(y,\theta). \tag{20}$$

If the preceding constraint is not valid, then we solve the equation for the phase θ. It has to be valid with

$$\theta = \cos^{-1}\left(-\sqrt{\frac{p(\neg y|x)\cdot p(\neg y|\neg x)}{p(y|x)\cdot p(y|\neg x)}}\cdot\cos(\theta_\neg)\right) \tag{21}$$

and the constraint

$$p(\neg y) \leq p(y) \tag{22}$$

simplified as

$$p(y,\theta) = 1-p(\neg y,\theta_\neg). \tag{23}$$

For equality in the constraint $p(y) = p(\neg y)$ both cases become equal.

Probability Waves are Smaller Equal One We assume without loss of generality

$$p(y) \leq p(\neg y) \tag{24}$$

it follows that

$$p(y) \leq 0.5. \tag{25}$$

By the inequality of arithmetic and geometric means

$$\sqrt{p(y,x) \cdot p(y,\neg x)} \leq \frac{p(y,x) + p(y,\neg x)}{2} \tag{26}$$

$$2 \cdot \sqrt{p(y,x) \cdot p(y,\neg x)} \leq p(y) = p(y,x) + p(y,\neg x) \tag{27}$$

$$p(y,\theta) = p(y) + 2 \cdot \sqrt{p(y|x) \cdot p(x) \cdot p(y|\neg x) \cdot p(\neg x)} \cdot \cos(\theta) \leq 2 \cdot p(y) \leq 1. \tag{28}$$

2.3 Probability Waves

Using the values of Table 1 and Table 2, we can determine the probability waves

$$p(\neg y, \theta_{\neg}) = p(\neg y) + 2 \cdot \sqrt{p(\neg y|x) \cdot p(x) \cdot p(\neg y|\neg x) \cdot p(\neg x)} \cdot \cos(\theta_{\neg}). \tag{29}$$

and

$$p(y,\theta) = p(y) + 2 \cdot \sqrt{p(y|x) \cdot p(x) \cdot p(y|\neg x) \cdot p(\neg x)} \cdot \cos(\theta). \tag{30}$$

as indicated in Figure 3. For

$$p(y) \leq p(\neg y) \tag{31}$$

the maximal interference is

$$\pm Inter_{max} = \pm\sqrt{p(y,x) \cdot p(y,\neg x)} \tag{32}$$

and for

$$p(\neg y) \leq p(y) \tag{33}$$

the maximal interference is

$$\pm Inter_{max} = \pm\sqrt{p(\neg y,x) \cdot p(\neg y,\neg x)}. \tag{34}$$

We can define the intervals that describe the probability waves as

$$I_y = [p(y) - Inter_{max}, p(y) + Inter_{max}] \tag{35}$$

and

$$I_{\neg y} = [p(\neg y) - Inter_{max}, p(\neg y) + Inter_{max}] \tag{36}$$

with

$$p(\neg y, \theta_{\neg}) \in I_{\neg y}, \quad p(y,\theta) \in I_y \tag{37}$$

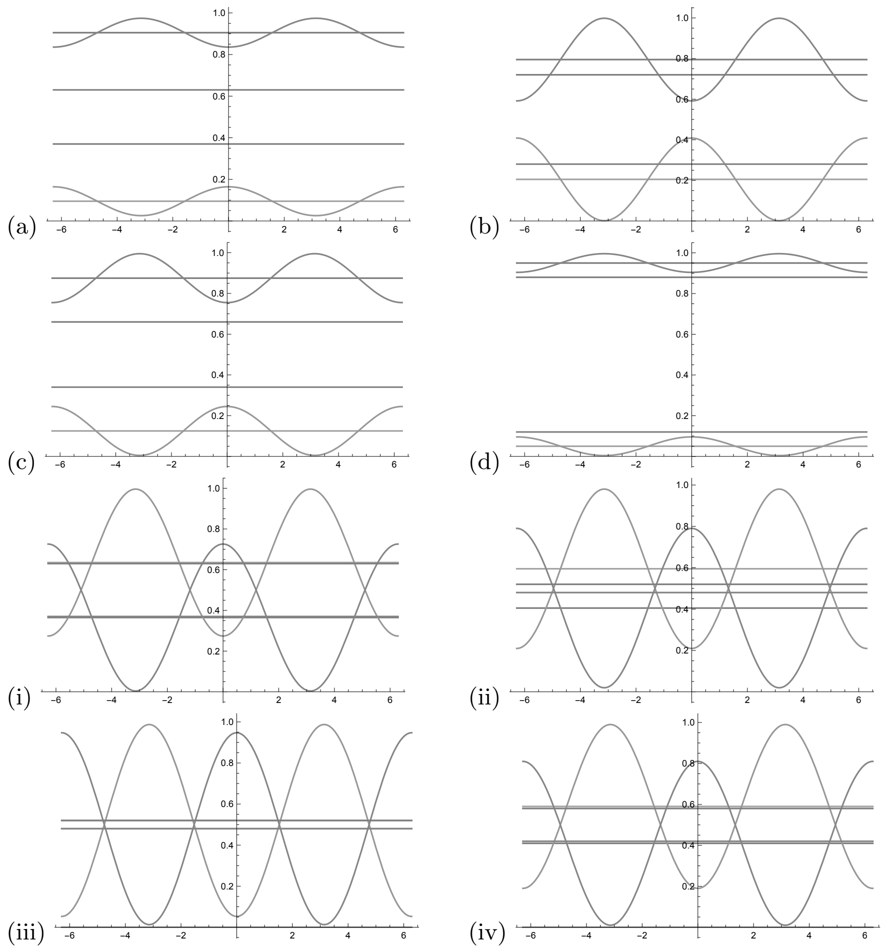

Fig. 3. Probability waves for the experiments described in Table 1 and Table 2. In plots (a) - (d) the waves $p(\neg y, \theta_{\neg})$ are around $p(\neg y)$, (for (e) see Figure 2). In the plots (i) - (iv) the waves $p(y, \theta)$ are around $p(y)$. Additionally the values $p_{sub}(\neg y)$ and $p_{sub}(y)$ are indicated by a line. Note that the curves in the plots (i) - (iv) overlap.

3 Law of Maximal Uncertainty

How can we choose the phase θ for a probability value that reflects the case of not knowing what the correct value is and without losing information about the probability waves? We answer this question by proposing the law of maximal uncertainty is based on two principles, the principle of entropy and the mirror principle.

3.1 Principle of Entropy

For the case in which both interval do not overlap

$$I_y \cap I_{\neg y} = \emptyset \tag{38}$$

the values of the waves that are closest to the equal distribution are chosen. By doing so the uncertainty is maximised and the information about the probability wave is not lost. The principle of maximum entropy states that the probability distribution which best represents the current state of knowledge is the one with largest entropy [11–13]. In the case of a binary event the highest entropy corresponds to an equal distribution

$$H = -p(y) \cdot \log_2 p(y) - p(\neg y) \cdot \log_2 p(\neg y) = -log_2 \cdot 0.5 = 1 \ bit. \tag{39}$$

For $p(y) \leq p(\neg y)$ the closest values to the the equal distribution are for $\theta = 0$, the ends of the interval for

$$p_q(y) = p(y) + 2 \cdot \sqrt{p(y,x) \cdot p(y,\neg x)} \approx 2 \cdot p(y) \tag{40}$$

and

$$p_q(\neg y) = 1 - p_q(y). \tag{41}$$

For $p(\neg y) \leq p(y)$ the closest values to the the equal distribution are

$$p(\neg y)_q = p(\neg y) + 2 \cdot \sqrt{p(\neg y,x) \cdot p(\neg y,\neg x)} \approx 2 \cdot p(\neg y) \tag{42}$$

and

$$p_q(y) = 1 - p_q(\neg y). \tag{43}$$

3.2 Mirror Principle

For the case where the intervals overlap

$$I_y \cap I_{\neg y} \neq \emptyset \tag{44}$$

an equal distribution maximises the uncertainty but loses the information about the probability wave. To avoid the loss we do not change the entropy of the system we use the positive interference as defined by the law of balance. When the intervals overlap the positive interference is approximately of the size of smaller probability value since the arithmetic and geometric means approach each other, see Equation 26. We increase uncertainty by mirror the "probability values". For the case $p(y) \leq p(\neg y)$ we assume

$$p_q(\neg y) = 2 \cdot \sqrt{p(y,x) \cdot p(y,\neg x)} \approx p(y) \tag{45}$$

and

$$p_q(y) = 1 - p_q(\neg y). \tag{46}$$

For $p(\neg y) \leq p(\neg y)$

$$p_q(y) = 2 \cdot \sqrt{p(\neg y,x) \cdot p(\neg y,\neg x)} \approx p(\neg y) \tag{47}$$

and

$$p_q(\neg y) = 1 - p_q(y). \tag{48}$$

Table 3 summarises the intervals that describe the probability waves, the resulting probabilities p_q that are based on the law of maximal uncertainty, the subjective probability and the classical probability values. We compare the re-

Table 3. Probability waves, the resulting probabilities p_q that are based on the law of maximal uncertainty, the subjective probability and the classical probability values. Entries (a)-(e) are based on the principle of entropy and entries (i)-(iv) are based mirror principle.

Experiment	$I_{\neg y}$	$p_{sub}(\neg y)$	$p_q(\neg y)$	$p(\neg y)$
(a)	[0.84, 0.97]	0.63	0.84	0.91
(b)	[0.59, 1]	0.72	0.59	0.79
(c)	[0.76,1]	0.66	076	0.88
(d)	[0.90, 1]	0.88	0.90	0.95
(e) Average	[0.77, 0.99]	0.72	0.77	0.88
Experiment	I_y	$p_{sub}(y)$	$p_q(y)$	$p(y)$
(i)	[0.27, 0.98]	0.37	0.36	0.64
(ii)	[0.20, 0.98]	0.48	0.39	0.59
(iii)	[0.09, 0.99]	0.41	0.45	0.54
(vi) Average	[0.19, 0.99]	0.42	0.40	0.59

sults that are based on probability waves and the law of maximal uncertainty with previous works that deal with predictive quantum-like models for decision making, see Table 4. The dynamic heuristic [18] is used quantum-like bayesian networks. Its parameters are determined by examples from a domain. On the other hand In the Quantum Prospect Decision Theory the values need not to be adapted to a domain. The quantum interference term is determined by the Interference Quarter Law. The quantum interference term of total probability is simply fixed to a value equal to 0.25 [22].

Table 4. Comparison between the Quantum Prospect Decision Theory (DT) [22] , the dynamic heuristic (DH) [18] and the law of maximal uncertainty (MU) of the balanced quantum-like model. The results of the dynamic heuristic (DH) and the law of maximal uncertainty (MU) are similar, however the the law of maximal uncertainty (MU) was not adapted to a domain.

Experiment	*observed*	*PDT*	*DH*	*MU*
(a)	0.63	0.65	0.64	0.84
(b)	0.72	0.54	0.71	0.59
(c)	0.66	0.63	0.80	0.76
(d)	0.88	0.70	0.90	0.90
(e) Average	0.72	0.63	0.76	0.77
(i)	0.37	0.39	0.36	0.36
(ii)	0.48	0.35	0.40	0.39
(iii)	0.41	0.29	0.41	0.45
(iv) Average	0.42	0.34	0.39	0.40

4 Conclusion

Physical experiments indicate that wave functions are present in the world [21]. They state that the size does not matter and that a very large number of atoms can be entangled [9, 1]. Clues from psychology also indicate that human cognition is based on quantum probability rather than the traditional probability theory as explained by Kolmogorov's axioms [8, 6, 7, 5]. This approach could lead to the conclusion that a wave function can be present at the macro scale of our daily life.

We introduce a balanced Bayesian quantum-like model that is based on probability waves. The law of maximum uncertainty indicates how to choose a possible phase value of the wave resulting in a meaningful probability value. The law of maximal uncertainty of the balanced quantum-like model is not static, meaningful and does not need to be adapted to a specific domain. The results obtained show that the model can make predictions regarding human decision-making with a meaningful interpretation.

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